Coordination and Multiplicity:
Global Games

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Motivation

Large class of problems in macroeconomics & finance with complementarities

- currency attacks
- bank runs
- herding in financial markets
- price setting in New Keynesian models

Problem $\Rightarrow$ complementarities often induce multiple equilibria

Problems with Multiple Equilibria

Problems with multiple equilibria

- weak predictions
- no comparative statics
- equilibrium selection

In particular, equilibrium notion assumes an extreme amount of coordination

$\Rightarrow$ Global games are a way to model (more realistic) situations where coordination is difficult

Two Players Example

Carlsson and van Damme (1993)

	α_2	β_2
α_1	$(0,\ 0)$	$(0,\ \theta - 1)$
β_1	$(\theta - 1,\ 0)$	$(\theta,\ \theta)$

When $\theta \in (0, 1)$ & complete information (Common Knowledge)

$\Rightarrow$ two pure strategy equilibria: (α_1, α_2) and (β_1, β_2)

Incomplete Information

We could be general, but let's use parametric assumptions for clarity

- Player i observes $x_i = \theta + \sigma\epsilon_i$, with $\epsilon_i \sim^{iid} N(0,1)$
- Prior over θ is "improper": $\theta \sim$ Uniform on $\mathbb{R}$

$\Rightarrow$ Posterior beliefs follow $\theta \mid x_i \sim N(x_i, \sigma^2)$

Now the payoff of action β is uncertain

Cases:

- If $\sigma \to 0$, common knowledge $\Rightarrow$ perfect coordination & multiple equilibria
- If $\sigma \to \infty$, common knowledge (& perfect coordination) again: $\mathbb{E}(\theta|x_i) = \mathbb{E}(\theta)$
  $\Rightarrow$ multiple equilibria if $\mathbb{E}(\theta) \in (0,1)$
- If $\sigma \in (0,\infty)$, we construct an eqm step by step

Strategic Uncertainty

If $\sigma \in (0, \infty)$, signal x_i matters and is not known by other players
$\Rightarrow$ *"strategic uncertainty"*

1) First iteration

- Suppose p1 believes p2 will play β for sure
- p1 plays β iff $\mathbb{E}(\theta|x_1) > 0$
- since $\mathbb{E}(\theta|x_1) = x_1$, p1 plays β iff $x_1 > \overline{x}^1 = 0$

Second Iteration

Unreasonable to expect p2 to play β for sure

2) Second iteration

- Suppose p2 believes p1 will play β when $x_1 > 0$
- Then p2 plays β when its expected payoff

$$[1 - P(\beta_1 \mid x_2)][\mathbb{E}(\theta|x_2) - 1] + P(\beta_1 \mid x_2)\mathbb{E}(\theta|x_2) > 0$$

$$x_2 - [1 - P(\beta_1 \mid x_2)] > 0$$

and

$$P(\beta_1 \mid x_2) = P(x_1 > 0 \mid x_2) = P(\theta + \sigma\epsilon_1 > 0 \mid x_2) = P(x_2 - \sigma\epsilon_2 + \sigma\epsilon_1 > 0)$$
$$= P\left(\epsilon_1 - \epsilon_2 > -x_2/\sigma\right) = 1 - \Phi\left(\frac{-x_2}{\sigma\sqrt{2}}\right)$$

since $(\epsilon_1 - \epsilon_2) \sim N(0, 2)$

- Hence, p2 plays β iff $x_2 > \Phi\left(\frac{-x_2}{\sigma\sqrt{2}}\right)$ $\rightarrow \overline{x}^2 > \overline{x}^1 = 0$

Keep going...

3) Third iteration

- Suppose p1 believes p2 will play β when $x_2 > \overline{x}^2$
- Then p1 plays β when its expected payoff

$$x_1 - [1 - P(\beta_2)] > 0$$

and

$$P(\beta_2 \mid x_1) = P(x_2 > \overline{x}^2 \mid x_1) = P(x_1 - \sigma\epsilon_1 + \sigma\epsilon_2 > \overline{x}^2) =$$
$$= P\left(\epsilon_2 - \epsilon_1 > (\overline{x}^2 - x_1)/\sigma\right) = 1 - \Phi\left(\frac{\overline{x}^2 - x_1}{\sigma\sqrt{2}}\right)$$

- Hence, p1 plays β iff $x_1 > \Phi\left(\frac{\overline{x}^2 - x_1}{\sigma\sqrt{2}}\right)$ $\rightarrow \overline{x}^3$
- and since

$$\Phi\left(\frac{\overline{x}^2 - x}{\sigma\sqrt{2}}\right) > \Phi\left(\frac{-x}{\sigma\sqrt{2}}\right) > 0$$

we know

$$\overline{x}^3 > \overline{x}^2 > \overline{x}^1 = 0$$

Solution

As we saw, the expected payoff of playing β for i, given j has threshold $\overline{x}_j$

$$A(x_i, \overline{x}_j) = x - \Phi\left(\frac{\overline{x}_j - x_i}{\sigma\sqrt{2}}\right)$$

We look for the *symmetric* signal threshold that makes agents indifferent

$$\overline{x} = \Phi\left(\frac{\overline{x} - \overline{x}}{\sigma\sqrt{2}}\right) \quad \Rightarrow \quad \overline{x} = \frac{1}{2}$$

Remarks

The (ex-ante) probability each player plays β is

$$P(x_i > 1/2) = P(\theta + \sigma\epsilon_i > 1/2) = P\left(\epsilon_i > \frac{\frac{1}{2} - \theta}{\sigma}\right) = \Phi\left(\frac{\theta - \frac{1}{2}}{\sigma}\right)$$

As $\sigma \to 0$, the equilibrium is $\begin{cases} (\alpha_1, \alpha_2) & \text{if } \theta < \frac{1}{2} \\ (\beta_1, \beta_2) & \text{if } \theta > \frac{1}{2} \end{cases}$

- a grain of doubt (on the others' action) gives us equilibrium selection
- given θ, agents' investment decisions are independent
- completely $\neq$ complete-info or no-info worlds, where agents coordinate perfectly

Continuum of Players

- So far, we considered a two-players game
- In macro and finance, we care more about a *continuum* of players

- We now consider a continuum of players
- We look at applications to bank runs and currency crises
- Agents have two options:
  - a safe action, constant payoff
  - a risky action, payoff depends on θ *and* what others do A
    (run to the bank, attack the currency)
- If enough agents take the risky action, something happens
  (currency crash, bank fails) $\rightarrow$ *"Global Games of Regime Change"*

In a nutshell

- **complementarities**: agents care about what others do (FOMO!), but...
- **strategic uncertainty**: agents are not sure about what others will do

Bank Runs

Goldstein and Pauzner (2005)

Demand–Deposit Contracts and the Probability of Bank Runs

- Continuum of measure 1 of depositors
- Must decide whether to withdraw their deposits early or "trust" the bank
- Deposits invested by the bank in a long term project with payoff θ
- Actions
  1. Early withdrawal ("run"): sure payoff of 0
     $A \in [0, 1]$ is the mass of early withdrawals agents
  2. Late withdrawal: payoff of $\theta - A$
     [Normalisation: can think of safe payoff as 1 and risky payoff as $1 + \theta - A$.]

As before, if all know that $\theta \in (0, 1) \implies$ multiple equilibria ($A = \{0, 1\}$)

Dispersed information

- Player i observes $x_i = \theta + \sigma\epsilon_i$, with $\epsilon_i \sim^{iid} N(0, 1)$
- Prior over θ is improper: $\theta \sim$ Uniform on $\mathbb{R}$
- Posterior beliefs then are $\theta \mid x_i \sim N(x_i, \sigma)$

Strategic Uncertainty

What about beliefs of other's actions?

- Suppose everyone follows a threshold strategy: withdraw if $x_i < \overline{x}$
- By the law of large numbers, the mass of people running on the bank = Prob(j runs)

$$A(\overline{x}) = P(x_j < \overline{x})$$

- Agent i's beliefs about others (or agent j's signal) is as before

$$\mathbb{E}[A \mid x_i] = P(x_j < \overline{x} \mid x_i) = P(x_i - \epsilon_i + \epsilon_j < \overline{x}) = \Phi\left(\frac{\overline{x} - x_i}{\sigma\sqrt{2}}\right)$$

- So the expected payoff of the risky action (not running on the bank) is

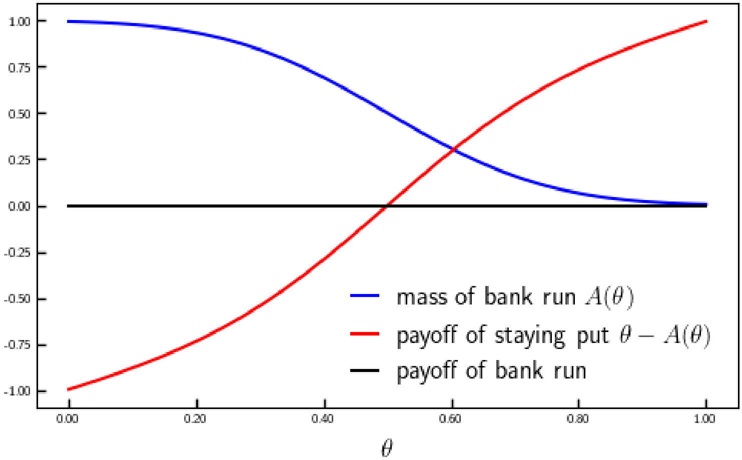
$$\mathbb{E}(\theta - A \mid x_i) = x_i - \Phi\left(\frac{\overline{x} - x_i}{\sigma\sqrt{2}}\right)$$

- Indifference condition

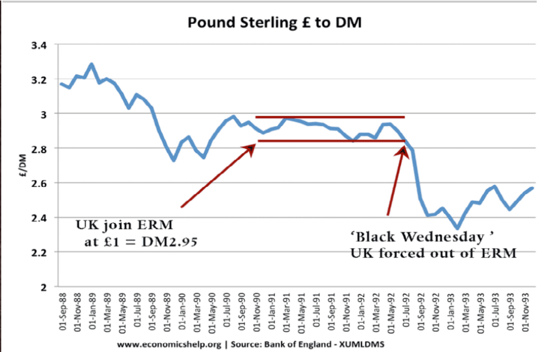
$$\mathbb{E}(\theta - A \mid \overline{x}) = 0 \quad \Longrightarrow \quad \overline{x} = \frac{1}{2}$$

$\Rightarrow$ same as before! 2-Players Eqm Alternative derivation

Equilibrium Illustration



Currency Crises (simplified)



Morris and Shin (1998)

Unique Equilibrium in a Model of Self-Fulfilling Currency Attacks

- Continuum of measure 1 of speculators/currency short-sellers
- Exchange rate peg, abandoned if central bank unable to sustain it
- "Not Attack" (stay put): payoff 0
- "Attack" (short-sell):
  - cost t of attack (transaction cost or cost of xccy funding)
  - payoff if crash (regime change) $1 - t$
  - payoff if resist (status quo) $-t$
- The central bank can defend the peg if
  - reserves > short-selling volume $\theta > A$

Common knowledge

- Once again, if all know $\theta \in (0, 1)$, there are multiple equilibria $A \in \{0, 1\}$

Dispersed Information

Assume a slightly richer information structure here

- Player i observes $x_i = \theta + \sigma_x \epsilon_i$, with $\epsilon_i \sim^{iid} N(0,1)$

- Normal prior over θ: $\theta \sim N(\mu_\theta, \sigma_\theta^2)$

- As usual, precisions are defined as $\tau_\theta := 1/\sigma_\theta^2, \tau_x := 1/\sigma_x^2$

- Posterior beliefs then are

$$\theta \mid x_i \sim N\left(\frac{\tau_\theta \mu_\theta + \tau_x x_i}{\tau_\theta + \tau_x}, \frac{1}{\tau_\theta + \tau_x}\right)$$

Finding the equilibrium

- Suppose agents attack iff $x_i < \overline{x}$
- Actual mass of attackers is given by

$$A(\theta) = \Phi\left(\frac{\overline{x} - \theta}{\sigma_x}\right)$$

- Critical level of fundamentals is defined as

$$\theta^* = \Phi\left(\frac{\overline{x} - \theta^*}{\sigma_x}\right)$$

  if θ higher (lower), peg maintained (abandoned)
- Expected payoff for agent with x_i who sells FX short

$$P(\theta < \theta^* \mid x_i)(1 - t) + P(\theta > \theta^* \mid x_i)(-t)$$

- Simplify and consider indifference condition of marginal agent with $x_i = \overline{x}$

$$1 - t = \Phi\left(\sqrt{\tau_\theta + \tau_x}\left(\theta^* - \frac{\tau_\theta \mu_\theta + \tau_x \overline{x}}{\tau_\theta + \tau_x}\right)\right)$$

Finding the Equilibrium

- The two equations

$$\theta^* = \Phi\left(\frac{\overline{x} - \theta^*}{\sigma_x}\right)$$

$$1 - t = \Phi\left(\sqrt{\tau_\theta + \tau_x}\left(\theta^* - \frac{\tau_\theta \mu_\theta + \tau_x \overline{x}}{\tau_\theta + \tau_x}\right)\right)$$

can be solved for $(\theta^*, \overline{x})$

- Combining them we get

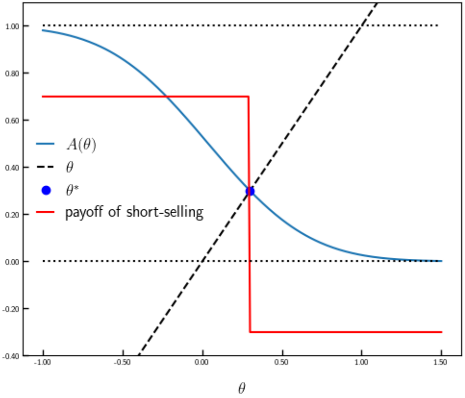
$$\theta^* = \Phi\left(\frac{\tau_\theta}{\sqrt{\tau_x}}(\theta^* - \mu_\theta) - \frac{\sqrt{\tau_\theta + \tau_x}}{\sqrt{\tau_x}}\Phi^{-1}(1 - t)\right)$$

and we get θ^* as a function of parameters

Special Case

In the special case of a uniform prior ($\tau_\theta/\sqrt{\tau_x} = 0$), we get the simple solution

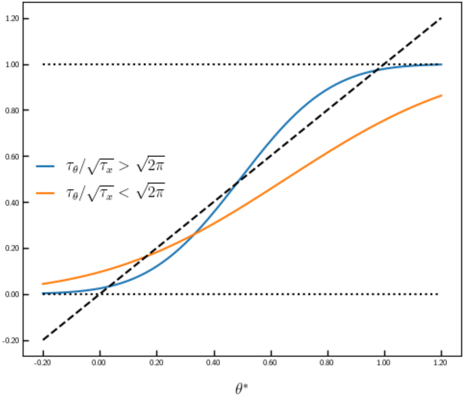
$$\theta^* = t; \quad \overline{x} = t + \sigma_x \Phi^{-1}(t)$$



Equilibrium Uniqueness

With $\tau_\theta/\sqrt{\tau_x} > 0$, we get uniqueness when private signal are relatively precise wrt prior/public signals, i.e. with sufficient amount of strategic uncertainty

If public signal more precise, coordination forces remain strong and multiplicity is possible



Two Types of Uncertainty

Fundamental uncertainty $\rightarrow$ on the payoff θ (τ_θ)
Strategic uncertainty $\rightarrow$ on the actions of others (τ_x)

Consider $\tau_x \rightarrow \infty$

- agents learn θ perfectly, no more fundamental uncertainty
- but what are their beliefs on the actions of others?
- we can ask what is the distribution of the *random variable* $A(\theta)$ for agent i

Strategic Uncertainty /1

"I observe signal x. What is the probability that the mass of agents attacking is smaller than z?"

That is

$$
\begin{aligned}
P\Big(\overbrace{P(x_j < \overline{x})}^{A} < z \mid x_i\Big) &= P\left(\Phi\left(\frac{\overline{x}-\theta}{\sigma_x}\right) < z \mid x_i\right) = \\
&= P\left(\theta > \overline{x} - \sigma_x \Phi^{-1}(z) \mid x_i\right) \\
&= P\left((\theta - m)/s > (\overline{x} - \sigma_x \Phi^{-1}(z) - m)/s\right) \\
&= 1 - \Phi\left(\sqrt{\tau_\theta + \tau_x}\left(\overline{x} - \frac{\tau_\theta \mu_\theta + \tau_x x_i}{\tau_x + \tau_\theta} - \frac{\Phi^{-1}(z)}{\sqrt{\tau_x}}\right)\right)
\end{aligned}
$$

and for the agent that has $x_i = \overline{x}$

$$
1 - \Phi\left(\frac{\tau_\theta}{\sqrt{\tau_x + \tau_\theta}}(\overline{x} - \mu_\theta) - \frac{\sqrt{\tau_\theta + \tau_x}}{\sqrt{\tau_x}}\Phi^{-1}(z)\right)
$$

Strategic Uncertainty /2

When

- $\tau_\theta = 0$ (diffuse prior)

  or

- $\tau_x \to \infty$ (no fundamental uncertainty)

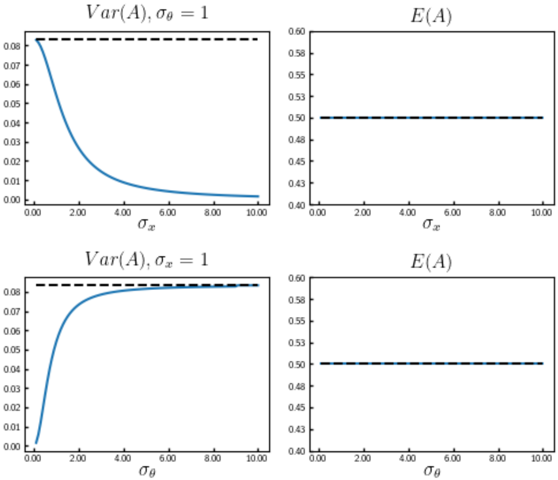
we get

$$P(A < z \mid x_i = \overline{x}) = z$$

$\Longrightarrow$ Agent $\overline{x}$ is "agnostic": assigns the same probability to any A!

[When $P(Y < z) = z$, Y is Uniform on the $[0, 1]$ interval.]

Strategic Uncertainty /3



References

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Appendix

Alternative Derivation of $\mathbb{E}(A(\theta) \mid x)$

$$\begin{aligned}
\mathbb{E}(A(\theta) \mid x) &= \int \Phi\left(\frac{k-\theta}{\sigma}\right) d\Phi\left(\frac{\theta - x}{\sigma}\right) \\
&= \int \Phi\left(\frac{k-x}{\sigma} - y\right) d\Phi(y) \\
&= \Phi\left(\frac{k-x}{\sigma\sqrt{2}}\right)
\end{aligned}$$

which is indeed the same as

$$P(x_j < k \mid x) = \Phi\left(\frac{k-x}{\sigma\sqrt{2}}\right)$$

Back